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2015

18th Annual High School Mathematical Contest in Modeling (HiMCM) Summary Sheet

(Please attach a copy of this page to your Solution Paper.)

Merits of the Later Zipper Merge

Team Control Number: 6057

Problem Chosen: A

Please paste or type a summary of your results on this page. Please remember not to include the name of your school, advisor, or team members on this page.

Merits of the Later Zipper Merge

Summary

Lane merge could induce potential road danger to our smooth and happy trips. How can we get through of the merge as soon as possible? How can we direct busy drivers to drive fairly? This is a very exciting and challenging topic to our mad math group. There are several different scenarios in lane merge: a) 2 lanes merge into 1; b) 3 lanes merge into 2 or 1; c) different speed limit. We crack the 3 scenarios with mathematic models one by one.

To address the 2-to-1 lane merge problem, we develop a "late zipper merge" strategy. By common practice, drivers merge into the left (main) lane as soon as they see the "lane closed" sign. Cars form a long queue in this lane and leave the right (merging) lane empty. Sometimes a "sneaky" driver drives in the empty lane and squishes in at the lane closure, making the "good and lawful" early mergers feel offended, since the through traffic has right-of-way. This may create delay, road rage, or accident.

In comparison to the "early merge" strategy, our "later zipper merge" is not only efficient but also fair, according to our math model. We recommend that drivers drive in both lanes until they're 300 feet ahead of the closure, then merge in alternating order.

In the first scenario, we use two models to compare these practices. Model 1 is a simplified and extreme case. Model 2 is a detailed and complex case. Both models yield the same conclusion that our "later zipper merge" has an advantage compared with "early merge."

In the second scenario, we analyze 3-to-2 and 3-to-1 lane merge cases. After sincere and hard attacks to the current practice (merge into left lane whenever possible), we come up with a perfectly calculated model. However, this ideal model is very hard to implement in real life since our drivers are not robots and are not likely to carry out exact orders. So we have to forfeit our attempt of perfection. The common practice is actually not as bad as it seems. Our only plea is to the highway designer: "Please, please do not design a highway where 3 lanes merge into 1 all at once. At least not in my hometown."

For the third scenario, we decided that the merge strategy is not very dependent on the speed limit, only on to how you change lanes. So there is not much difference in your actions if you are on a secondary road of 35 mph, verses on a highway of 65 mph. Since when driving we need about 3 seconds' distance ahead for precaution, we need to merge with a safe space between our car and the closure. If your speed is 65 mph, at about 300 feet ahead of the closure right-lane drivers adjust to the speed of the left lane and prepare to merge. If you are driving at 35 mph, the point where you change lanes will be 168 feet from the merge point. In addition, since the safe distance will be shorter, it'll be slightly harder to squish into the left lane.

In conclusion, we recommend that the DMV change our highway laws and driver education materials accordingly. We have been very happy to explore this simple yet profound problem. It's the reason we are mad about math.

CONTENTS

1 Introduction

When drivers see "right lane closed ahead" or "lane merge" signs on a highway, most of them would move into the left lane fairly quickly since the through traffic has the right-of-way according to laws of most US jurisdiction. We call it an "Early merge" strategy.

This practice leads to a common phenomenon in which cars form a long line in the left (main) lane and leave the right (merging) lane empty. Some bold drivers seize this chance. They drive in the empty right lane, pass many "good and lawful" early drivers and squish in at the very last moment at the merge point. Oh! The injustice in this! The already merged drivers might get irritated and blow their horns. This is only one side of the card – road rage. Even worse, they may insist on their lawful rights and block the right lane drivers from merging. This might lead to collision – the other side of the card. The right lane driver might also earn a "failure to yield" or "unsafe merge" citation because it is always their fault in our current law system.

2 Problem Restatement

- Analyze the effects of several methods of merging at a 2-to-1 lane closure;
- Find the fairest and most efficient driver actions for merging into a lane;
- Find what drivers should do if a 3-lane highway is reducing to two lanes or one lane;
- Discuss differences in fair and efficient driver behavior for merging on a 35 mph speed limit highway versus a 65 mph road;
- Develop guidelines for the DMV on driver education and laws and write a letter to that effect.

3 Breaking Down the Problem

The situation could be improved if we adapted a "later zipper merge" strategy. When you see a "merge" sign, you determine whether the traffic is heavy. If you are the only one on a major highway, it doesn't matter where you drive (obviously). The "zipper merge" strategy shows its advantage on a busy road. The gist:

1) You stay in your own lane and maintain your current speed for as long as possible, no matter whether you are in the left or right lane.

2) At about 300 hundred feet ahead of the merge point, right-lane drivers adjust to the speed of the left lane and prepare to merge.

3) Drivers leave room for the car in the adjacent lane to merge.

4) The drivers in either lane take turns to enter the through lane like teeth on a zipper while maintaining the current speed.

Later on, we will address our reasoning in details: I. why merge later instead of merging early II. why use zipper merge instead of the “right-of-way” III. why merge 300 feet ahead of the merge point

Our paper will address these questions in four parts:

1. Why the “later and zipper merge” strategy is more efficient and fair than the current “early merge” strategy; whether is there any difference between fair and efficient practices.

2. What drivers should do if a three lane highway is reducing to two lanes; how driver behavior should change if the three lanes merge into one lane.

3. Whether any behavior should change if drivers are on a second road of 35 mph verses on a highway of 65 mph.

4. Guidelines for DMV driver education material and change in road signs and law.

4 Assumptions and Justifications

4.1 Highway Assumptions

Assumption 1: A busy road refers to the cars driving in such a manner so that the distance between cars is exactly the safe distance: 3 seconds.

Justification: When a road is busy, cars will be very close together. However, in America most drivers are encouraged to leave at least 3 seconds’ distance in front of them to allow sufficient time in case the car in front suddenly stops.[2] Drivers always voluntarily leave more or less 3s for safety. We assume they leave in exactly 3s, for simplicity.

Assumption 2: The 2 (or 3) lanes have the same numbers of cars.

Justification: Since all lanes are busy, the lanes would have the same numbers of cars.

4.2 Vehicle Assumptions

Assumption 3: All the cars have the same length, 14 feet.

Justification: It is difficult and unnecessary to take into account the differences in individual vehicles. We found that most cars are about 14ft long. [3]

4.3 Driver Action Assumptions

Assumption 4: The merge sign is put at 1 mile before merge point and all of the drivers see the sign.

Justification: This assumption is needed for many calculations (we assume that all drivers leave in 3 seconds again).

Assumption5: drivers can get to exact speeds and distances.

Justification: That's what I would do. This is for simplicity.

Assumption 6: Ignore the difference while driving in curve or straight line.

Justification: Drivers should see the sign at about this point, and it is unnecessary to make complications.

Assumption7: The difference between driving in a straight line and merging can be neglected in the forward direction, and doesn't take any time.

Justification: Merging time is short and doesn't affect calculations much.

Assumption 8: Whenever a driver merges, he undergoes the following steps:

i) He flashes his signal light, the driver on the left sees instantly, and the car on the left, on the right, and cars behind this car begin to slow down.
ii) After 5s the left lane has just enough space to hold an additional car (including the safe distance) and the right-lane car merges.
iii) All cars return to the maximum speed (speed limit) as soon as possible.

Justification: This is required for our model. It is a reasonable method of merging.

Assumption 9: No cars in a left lane will merge into a right lane.

Justification: This doesn't happen much and we know it can't be an optimum method of merging lanes, as the efficiency would go down.

4.4 Other Assumptions

Assumption 10: "Fair" means the difference between average speeds of cars of cars is small.

Justification: If cars' average speeds are the same, they take the same amount of time to pass the merge and therefore it is fair.

Assumption 11: "Efficient" means the space on the highway is used most efficiently.

Justification: We assumed previously that speeds of cars after the closure is invariably the speed limit. So if there are more cars on a given space, more cars pass in a given time, and the merge is more efficient.

5 Symbols and Terms

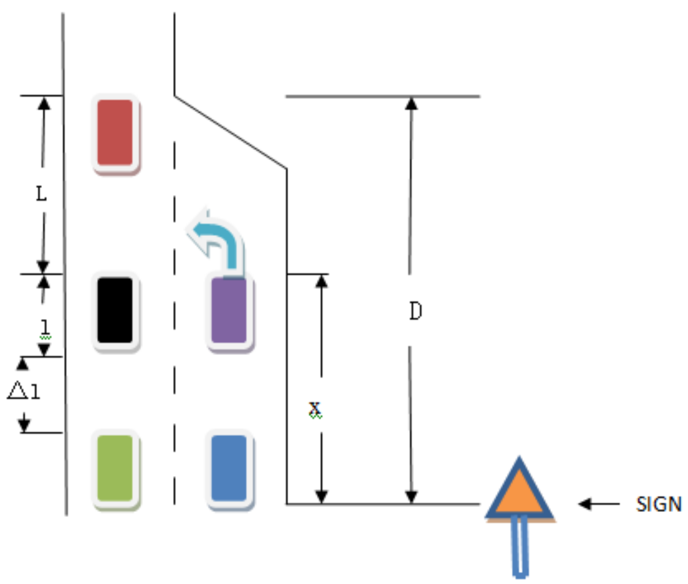


Figure 5.1: variables

Table 5.1: The Symbol Definition and Description

Symbol	Meaning
l	Length of each car
Δl	Safe distance between cars
L	Equivalent length of cars
x	The distance from sign to merging site
x_i	The distance from sign to merging site in lane i (i=1,2,3)
D	The distance from sign to closure
T	Amount of time passed
Δt	Distance between cars in seconds (3s)
t_1	Time used in first stage of merge
t_2	Time used in second stage of merge
t_3	Time used in third stage of merge
t_{in}	Total time used to merge in a lane
v_0	Speed limit or initial speed of cars
$v_{\min}$	Lowest speed during merge
v_i	Average speed in lane i (i=1,2,3)
$\lvert \Delta v_{ij} \rvert$	The difference between average speeds of lanes i and j (i, j =1,2,3)
$a_{\max}$	The maximum acceleration of a car
Q_i	The percentage of used space within a lane i (i=1,2,3)
Q	The percentage of used space on the segment of the highway
θ_i	Representation of lane i in the whole highway (i=1,2,3)

ρ_i	Density of cars in lane i (i=1,2,3)
P_{ij}	Probability function that describes the probability of merging from lane i to lane j at a given point x (i, j=1,2,3)
P_z	The equivalent probability of zipper-merging in the 3-to-2 merge

6 The First Scenario: Two-Lane Merge

6.1 Model 1: The Extreme Case

The following is a simplified version of the model. It does not use the assumptions or variables listed above.

Let's assume the cars act like robots and will carry out exact actions we tell them to. And the cars see the merge sign at 1 mile from the closure.

The first case: the cars see the "merge" sign, begin to slow down, and merge into the through lane at the closure, as shown in the figure 6.1:

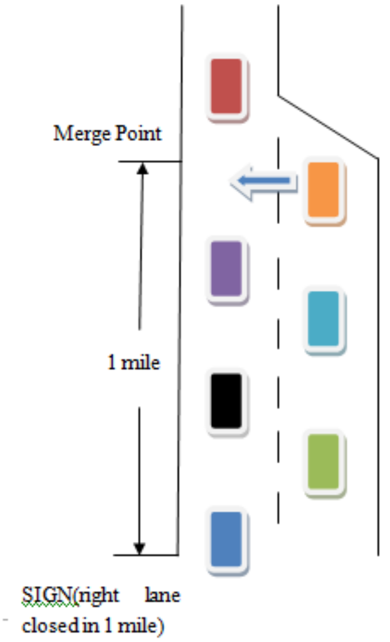


Figure 6.1: the extreme case

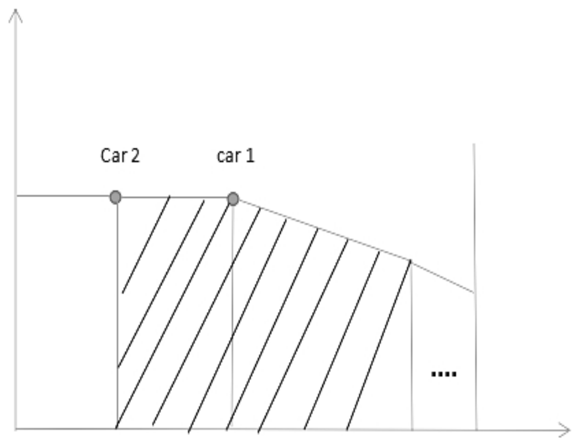


Figure 6.2: change in speed of the extreme case

The distance between two cars in the left lane changed from 300 feet to x feet. x is the 3-second distance with the new speed plus one car length (which we assume is 14 feet) and v_1 is the original speed 65 miles/hour (95.3 feet/second). If we assume that the new speed is v_2 feet/second, then

$$(95.3 - v_2) \times 3 = v_2 \times 3 + 14 \tag{6.1}$$

$$v_2 = 45.17 \text{ feet/second=30.8 mile/hour} \tag{6.2}$$

To finish this 1 mile, the average time cars need equals $\frac{1}{2}(v_1 + v_2)$, so:

$$\frac{1}{2}(v_1 + v_2) \times t = 5280 \qquad t\text{=75.4 seconds} \tag{6.3}$$

If the cars did not have to merge, the time needed to finish 1 mile is 5280/95.3=55.5 seconds. So the time delayed is 19.9 seconds.

The second case: the cars merge immediately when drivers see the "merge" sign. Assume the cars can change speed instantly to simplify our analysis. Cars on the right merge into the left lane without using any time. The cars in the left lane slow down to 45.17 feet/second to leave room for the cars on the right lane. The time the cars need to finish this one mile is:

$$\frac{5280}{95.3} = 55.5 \text{ seconds} \tag{6.4}$$

In the early merge case, the cars need 115.5 seconds to finish this one mile while in the 1st case the cars need 75.4 seconds to finish the same one mile. The 1st case is 115.5-75.4=40.1 seconds faster than the 2nd case for average cars.

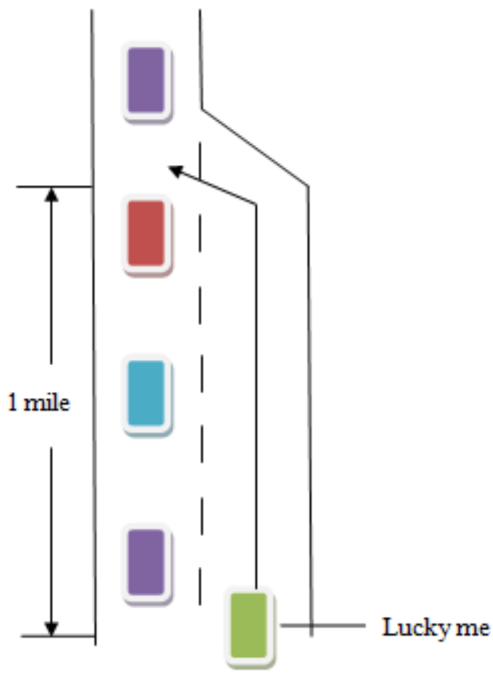


Figure 6.3: The third case of model 1

The third case is an extension from the 2nd case: Most cars in the right lane had merged at the 1 mile point while the last car in the right lane uses the empty right lane and drives to the closure. The time used to travel this one mile for this lucky car is 5280/95.3 = 55.4 seconds. All the cars in the left lane have to wait to make room for this lucky car. Their waiting time is [(45.17)(3) + 14] / 45.17 = 3.4 seconds. So the total time all the unlucky cars in the left lane need to travel this 1 mile is 115.5 + 3.4 = 119.9 seconds. No wonder drivers that merge early get angry! They spend 119.9 – 55.4 =64.5 second more than the last car, which took a short-cut. They also spend 119.9 – 75.4 = 44.5 seconds more than cars the in 1st case.

Our conclusion: the "later merge" strategy (or "zipper merge") is more efficient and fair than "early merge."

6.2 Model 2: A More Comprehensive Approach

6.2.1 Step I—why merge later instead of merging early

In the following we introduce a probability function and optimize the percentage of space used by cars and the difference in average speed of the lanes, respectively, to represent the efficiency and fairness, also respectively.

The detailed calculations are as follows:

We have the length of cars, l , the safe distance between cars Δl, the initial speed v_0 ,the equivalent length of cars L. They are related with the following equations:

$$L=l+\Delta l \tag{6.5}$$

$$\Delta l = \frac{v \cdot \Delta t}{2} = v \cdot (1.5s) \quad \text{(where } \Delta t \text{ equals 3s)}$$

The change in speed during a merge can be represented with the following graph:

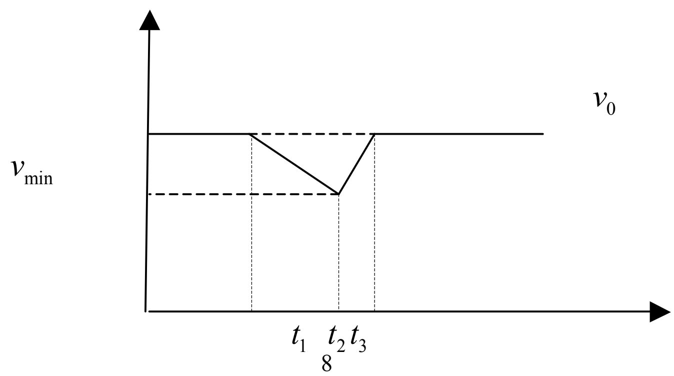


Figure 6.4: change in speed during merge

The time needed in each stage is as follows:

$$t_1 = 5s, \quad t_2 = 0s, \quad t_3 = \frac{v_0 - v_{\min}}{a_{\max}} \tag{6.6}$$

$$t_{in} = t_1 + t_2 + t_3 \tag{6.7}$$

We can see from the model that merging only affects the cars behind the merged car, not the cars in front of it, since the cars behind need to slow down to adjust to the new safe distance. According to this model, where the car merges has nothing to do with the amount of time used to pass the merge site. According to experience, however, most drivers merge early to avoid road rage which is most pronounced near the lane closure. The model above is incomplete.

To take into account how easy it is to merge, we define a probability function P. It has the following properties:

1. It decreases as x increases, where x is the distance from the sign and merging site, and $x \in [0, D]$.
2. It decreases as the number of cars that merge in increases.
3. It is inversely related to the density of cars in the merged lane.
4. It is directly related to the average velocity of cars in the merged lane.
5. When $x = 0$, $P = \frac{1}{2}$.

Also, we know that the three variables mentioned above are also related: the merged lane's car density is directly related to the number of cars and inversely related to the average speed of cars. Therefore we only need one of these to describe the probability function. If we choose the car density in the left lane, we have

$$P_{ij} = \frac{1}{1 + e^{k_1 x \rho_j}} \tag{6.8}$$

Where i is the merging lane and j is the merged lane, and k_1 is unknown.

To further simplify the model, we let the car density ρ_j be part of the unknown, and

$$P_{ij} = \frac{1}{1 + e^{k_1 x}} \tag{6.9}$$

The value of k_1 can be determined with the integral equation $\int_0^D \frac{1}{1+e^{k_1 x}} dx = 1$ since the sum of probabilities is 1.

The values of k_1 for when D is 1 mile, 0.5 miles, and 500ft are calculated as in table 2:

Table 2: evaluating k_1

D	D/m	k_1
1mile	1609.344	0.69314718
0.5mile	804.672	0.69314718
500ft	152.400	0.69314718

So we let k_1 be 0.6931.

Interpretation of the probability function:

P_{ij} is the probability that a driver is able to merge at a given point from lane i to lane j. Which is to say, P_{ij} is the percentage of cars that intend to and do merge at point x after seeing the sign, out of the total number of cars in the merging lane. Therefore at any given point x, a percentage P of cars change lanes in a period of time T and no longer affect their own lane, and the other cars this lane continue to search for an opportunity to merge.

Note that the following analysis will always be in a period of time T. We assume that the change in the flow of cars will be smooth and continuous. Therefore we may use averages to describe situations within this period of time.

For $Q_i (i = 1,2,3)$, the percentage of used space in lane i, we have

$$Q = \sum_i Q_i \theta_i \tag{6.10}$$

Where θ_i is the representation of a lane in the highway ($\theta_i = \frac{1}{2}$ in this case), $Q_i = \frac{\sum L}{D}$, and $v_i (i = 1,2,3)$ is the average speed in lane i and $\rho_i (i = 1,2,3)$ is the density of cars in lane i.

These two variables are related with the Greenshields linear equation $v_i = a + b\rho_i$. To find the values of the unknowns:

1. When $\rho_i = 0$ we have $v_i = v_0$, so $a = v_0$.

2. As $\rho_i \to \infty$, or $\rho_i = \rho_{\max} = \frac{D}{l}\frac{1}{D} = \frac{1}{l}$, we have $v_i = 0$, so $b = -\frac{v_0}{\rho_{\max}} = -v_0 l$.

Therefore

$$v_i = v_0(1 - \rho_i l) \tag{6.11}$$

Next we analyze the effects of merging at different places.

We use the following indications:

1. The percentage of space used by cars, Q

2. As stated in the assumptions showed previously, Q is an indication of efficiency since the terminal speed after the closure is constant.

$$Q = \sum_i Q_i\theta_i = Q_1 \times \frac{1}{2} + Q_2 \times \frac{1}{2} = 0.5 + \frac{x}{2D}(0 \le x \le D) \tag{6.12}$$

From this equation we see that Q and x are directly related to each other. Therefore the later the driver merges, the greater the the percentage of road used by cars, the more efficient the merge is.

3. The difference between the average speed of the two lanes, $|\Delta v_{12}|$

The smaller the difference, the fairer the merge.

$$v_1 = v_0(1 - \rho_1 l) \tag{6.13}$$

$$v_2 = v_0(1 - \rho_2 l) \tag{6.14}$$

So $$|\Delta v_{12}| = |v_0 l(\rho_1 - \rho_2)| = v_0 l\,|\Delta\rho_{12}| \tag{6.15}$$

Where $\rho_1 = \rho_0(1 + P_{21})$, $\rho_2 = \rho_0(1 - P_{21})$, $\rho_0 = \frac{D}{L_0}\frac{1}{D} = \frac{1}{L_0}$.

Therefore $|\Delta v_{12}| = v_0 l\,|\Delta\rho_{12}| = 2v_0 l\rho_0 P_{21}$

We can see that $|\Delta v_{12}|$ and P_{21} have a same trend with respect to x. When D is 30 meters, we get the following graph:

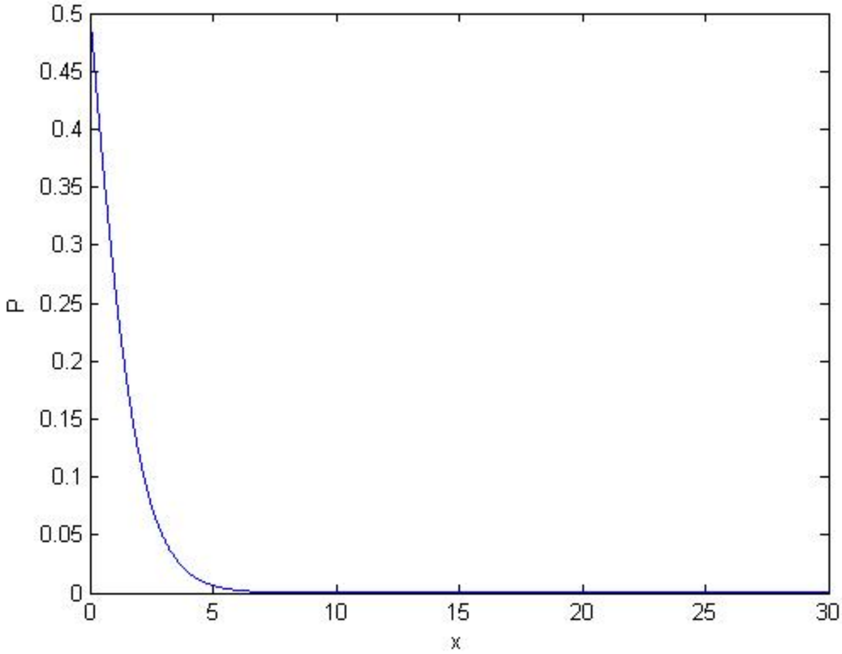


Figure 6.5: relationship between P_{21} and x

We see that P_{21} decreases as x increases, so $|\Delta v_{12}|$ also decreases as x increases.

Conclusion: a later merge (where x is at maximum D) is both fairest and most efficient. The later we carry out the merge, the better.

6.2.2 Step II—why use "zipper merge" instead of the "right-of-way"

If we use the zipper merge, the cars in the left and right lanes will take turns to merge, and the delayed time is minimized. However, if the left lane has the right-of-way and the cars in the left lane are unwilling to yield to the right, the cars in the right lane would have to wait until a car is willing to yield. Their waiting time would increase. Therefore the later merge is less efficient.

With the zipper merge, $P_{21} \equiv \frac{1}{2}$, and it is still fairest and most efficient at $x = D$. However in this case we have $|\Delta v_{12}| = 0$, whereas for the regular merge when $x = D$ we have $|\Delta v_{12}| > 0$ for all cases. So the zipper merge is fairer than the later merge.

What if before the zipper merge at the closure, some cars merge early? Well, previously it was demonstrated that early merges reduce the efficiency and fairness, so it would be better to have only one merge site, near the closure.

We can conclude that the zipper merge is better, both more efficient and fairer.

6.2.3 Step III—why merge 300 feet ahead of merge point

The “300 feet” dictation is merely for the majority of cases, where the speed limit is around 65mph. The general equation is 4.4v+14, where v is the speed limit in miles per hour. The value of the equation is in feet.

The explanation is as follows:

Most areas of the United States require drivers to stay 3 seconds behind the car in front. This is to allow the driver to have sufficient time to react if the previous driver should suddenly stop. Similarly, when the driver is preparing to merge, he should have at least 3 seconds ahead of him clear in case the unexpected happens. When the speed limit is 65mph, the distance is

$$(65mi/h)(3s)(5280ft/mi)(\frac{1}{3600}h/s)+14ft=300ft \tag{6.16}$$

As the length of the car is 14 feet. When the limit is 35mph, the distance is 168ft or approximately 170ft. Generally the distance is ($4.4v$+14) ft.

This equation may not be apt to be used among regular drivers, so we may decide on 300ft for most roads since most roads have a speed limit of 65. For smaller or larger roads, the distance is different.

7 Second Scenario: 3-Lane Merge

7.1 Verbal Analysis

If three lanes merge into two lanes, the situation is totally different from the case where 2 lanes merge into one.

On the two-lane merge case, all the cars of the right lane will merge into the left lane. It is simple to tell who goes to the merge point earlier and there is no need to keep balance.

The case becomes complex when 3 lanes merge into 2. We assume there are 3 lanes:

A, B, and C, where C is closed ahead. The road is fairly busy. We minimize our car number into 300 which means 100 cars in lane A and 100 cars in lane B and 100 cars in lane C. If only the cars in lane B and lane C use the late zipper merge strategy, the number of cars in lane A will be 100 and the number of cars in lane B will be 200. Lane B will be jammed. The situation would be like this:

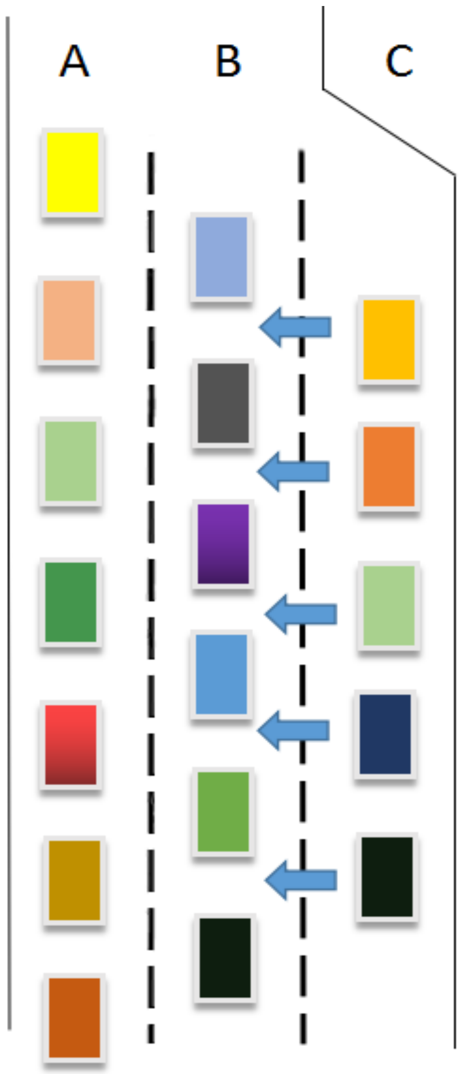


Figure 6.6: 3-to-2 merge

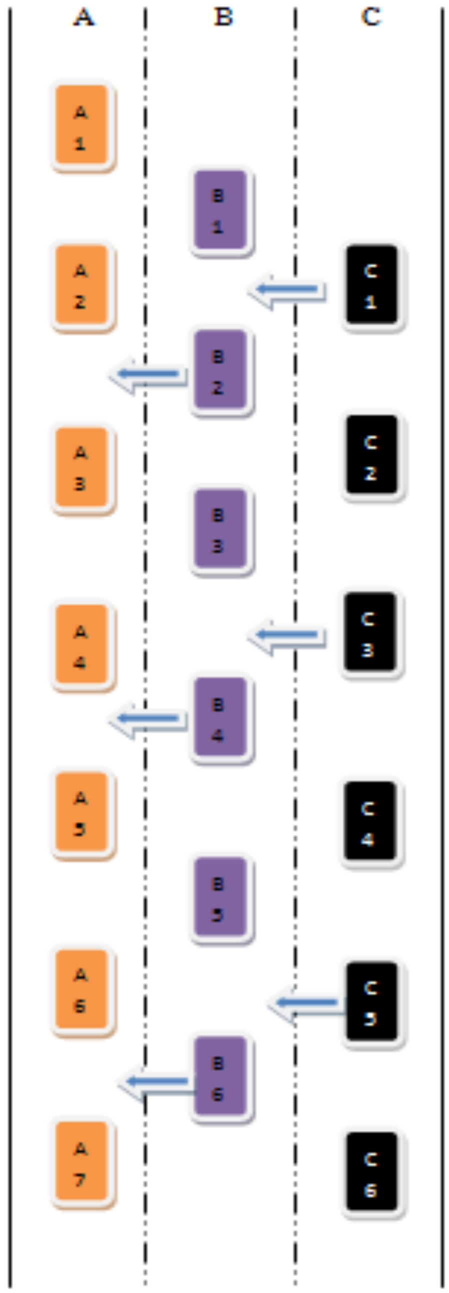


Figure 6.7: optima3-to-2merge

The best solution would be: 50 cars in lane B will have to change into lane A; 50 cars in lane B will stay in their lane and make room for the lanes in the cars in lane C; all of the cars in lane C will move into Lane B.

Please see the graph figure 6.7:

After the merge, the left lane will have 150 cars and the right lane will also have 150 cars. The number of cars in either lane is balanced and the cars end up with minimum waiting time.

Of course we would like this ideal situation. However, it is very hard to pinpoint which car is A1 or which car is B1 in a real-world driving situation. If only the cars were robots! So we recommend another approach to this problem:

1. If you are in lane A, stay in your current lane and make room for the cars in lane B that want to squish into your lane.

2. If you are in lane B and you merge into lane A if you get a chance. Otherwise you stay in your current lane.

3. If you are in lane C, you squish into lane A soon as you see a chance.

4. The drivers in lane C that still haven't merged at the closure use the zipper merge with lane B (this is not supposed to happen much).

This is the strategy our current drivers already use. So we recommend that we stay put and don't make any change.

Should we use the zipper merge if 3 lanes merge into 1? The good news is, there is only one lane left and we do not have to consider balance between 2 lanes. The bad news is, it is very hard to smoothly merge 3 cars into 1 lane at the very last moment. Plus, it is hard to distinguish which car comes to the merge point first. The crowd situation provokes delay, rage, or crash. So we do not recommend a "later zipper merge" in this situation. Use common sense instead.

Our previous strategy works just fine. The through traffic has right-of-way and we squish into the trough traffic as soon as possible. Question? Why do the police close two lanes at the same time? Please do not let me encounter this situation when I go on a vacation.

7.2 Mathematical Model

7.2.1 The 3-to-2 merge: regular merge model

The "regular merge model" refers to the situation where drivers merge randomly.

In this case, the third lane is merged and we have

$$P_{21} \equiv \frac{1}{2} \tag{6.17}$$

$$P_{32} = \frac{1}{1+e^{k_1 x_3}} \tag{6.18}$$

Which means that when drivers merge from lane 3 to lane 2, the model is similar to that of Model 2, Step I (P_{21}), while both lanes 2 and 1 are through, which is why P_{21} is constant ($\frac{1}{2}$).

Now we can optimize the efficiency and fairness respectively.

1. The percentage of space used by cars, Q

$$Q = Q_1 + Q_2 + Q_3 = 1\times\frac{1}{3} + 1\times\frac{1}{3} + \frac{x_3}{D}\times\frac{1}{3} = \frac{2}{3} + \frac{x_3}{3D} \quad (0 \le x_3 \le D) \tag{6.19}$$

We can see that Q increases as x_3 increases, so efficiency is maximized at $x_3 = D$.

2. The difference between the average speed of the two lanes, $|\Delta v_{12}|$

Here we have

$$\rho_3 = \rho_0(1-P_{32}) \tag{6.20}$$

$$\rho_2 = \rho_0(1+P_{32})(1-P_{21}) = \rho_0(\frac{1}{2}+\frac{1}{2}P_{32}) \tag{6.21}$$

$$\rho_1 = \rho_0(1+(1+P_{32})P_{21}) = \rho_0(\frac{3}{2}+\frac{1}{2}P_{32}) \tag{6.22}$$

Where $\rho_0 = \frac{D}{L_0}\frac{1}{D} = \frac{1}{L_0}$.

Therefore

$$|\Delta\rho_{12}| = |\rho_1 - \rho_2| = \rho_0 \tag{6.23}$$

$$|\Delta\rho_{23}| = |\rho_2 - \rho_3| = |\rho_0(\frac{3}{2}P_{32} - \frac{1}{2})| \tag{6.24}$$

$$|\Delta\rho_{13}|=|\rho_1-\rho_3|=|\rho_0(\frac{3}{2}P_{32}+\frac{1}{2})| \tag{6.25}$$

We can see that the minimum of $|\Delta\rho_{23}|$ occurs at $P_{32}=\frac{1}{3}$ and is 0. $|\Delta\rho_{12}|$ is unrelated to x_3 and $|\Delta\rho_{13}|$ decreases with the increase of x_3. To let the merge be fair, we let $P_{32}=\frac{1}{3}$. So we have figure 6.8(derived from figure 6.5):

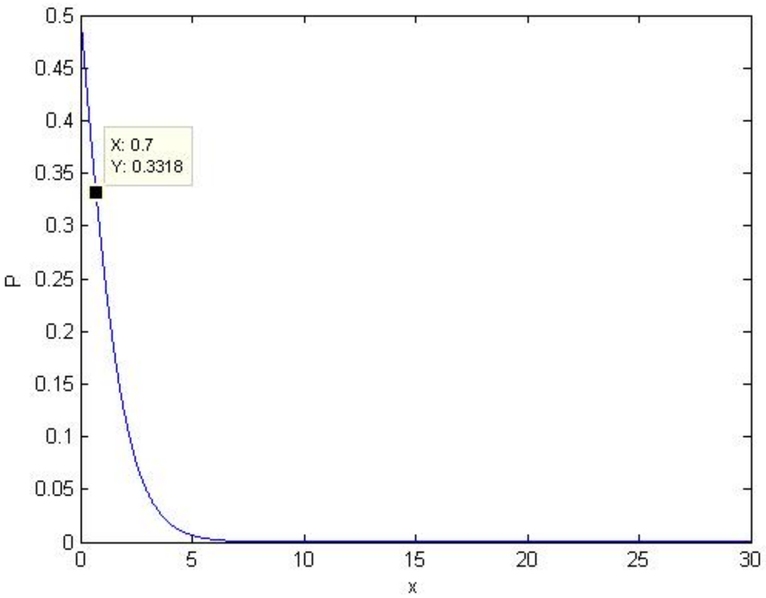


Figure 6.8: relationship between P_{32} and x , with P_{32}=1/3

Here we can see that $x_3 \approx \frac{0.7}{30}D = 0.0233D$, so the driver should merge $0.0233D$ from the sign to make it fairest.

Comparing the two analyses, we see we have a problem: it is most efficient near the closure, but fairest at $0.0233D$ from the sign (which is very far from the closure).

7.2.2 The 3-to-2 merge: proposed merge model

To solve this problem, we propose the following:

We allow a number of cars to merge early and let the rest zipper-merge near the closure. We can see that the percentage of space used (Q) is highest in this case, and therefore so is the efficiency, so let us now address the issue of fairness.

We let $P_z = \frac{1}{2}$ be the equivalent probability of zipper-merging.

The following is similar to our previous model of the regular merge model:

$$\rho_3 = \rho_0(1 - P_{32})P_z = \rho_0(\frac{1}{2} - \frac{1}{2}P_{32}) \tag{6.26}$$

$$\rho_2 = \rho_0(1 + P_{32})(1 - P_{21})P_z = \rho_0(\frac{1}{4} + \frac{1}{4}P_{32}) \tag{6.27}$$

$$\rho_1 = \rho_0(1 + (1 + P_{32})P_{21}) = \rho_0(\frac{3}{2} + \frac{1}{2}P_{32}) \tag{6.28}$$

Where $\rho_0 = \frac{D}{L_0}\frac{1}{D} = \frac{1}{L_0}$.

And accordingly

$$|\Delta\rho_{12}| = |\rho_1 - \rho_2| = |\rho_0(\frac{5}{4} + \frac{1}{4}P_{32})| \tag{6.29}$$

$$|\Delta\rho_{23}| = |\rho_2 - \rho_3| = |\rho_0(-\frac{3}{4}P_{32} + \frac{1}{4})| \tag{6.30}$$

$$|\Delta\rho_{13}| = |\rho_1 - \rho_3| = |\rho_0(P_{32} + 1)| \tag{6.31}$$

To promote fairness, we again let $P_{32} = \frac{1}{3}$, so $x_3 = 0.0233D$.

The analysis above gives us the following suggestion:

A third of the cars in lane 3 merge at 0.0233*D* the rest zipper-merge close to the lane closure. The cars in lane 2 merge to lane 1 as soon as they get the chance (in theory where they merge has no effect but in reality it is recommended that they merge early), and if they don't, they zipper-merge with lane 3. However, it is rather difficult to implement this strategy as the way it is, since the reality may be more complex.

Drivers in lanes 2 or 3 merge left whenever possible, and drivers in lanes 1 or 2 make way for them. At the closure any unmerged cars zipper-merge left. This method is both fair and efficient, and is consistent with the verbal analysis.

7.2.3 The 3-to-1 merge: regular merge model

Here the second and third lanes are merged and for the probability function we have

$$P_{21} = \frac{1}{1 + e^{k_2 x_2}} \tag{6.32}$$

$$P_{32} = \frac{1}{1+e^{k_3 x_3}} \tag{6.33}$$

Which means that when drivers merge from lane 3 to lane 2, or from lane 2 to lane 3, the model is also similar to that of Model 2, Step I (P_{21}). Therefore the probability function doesn't change.

We optimize the efficiency and fairness respectively again.

1. The percentage of space used by cars, Q

$$Q = Q_1 + Q_2 + Q_3 = 1\times\frac{1}{3} + \frac{x_2}{D}\times\frac{1}{3} + \frac{x_3}{D}\times\frac{1}{3} = \frac{1}{3} + \frac{x_2}{3D} + \frac{x_3}{3D} \qquad (0 \le x_2 \le x_3 \le D) \tag{6.34}$$

Note that when $x = D$, we have $x_2{=}x_3$.

We see that Q is directly related to x_2 and x_3, so the merge is most efficient when $x_2 = x_3 = D$.

2. The difference between the average speed of the two lanes, $|\Delta v_{12}|$

Again, we have

$$\rho_3{=}\rho_0(1-P_{32}) \tag{6.35}$$

$$\rho_2{=}\rho_0(1+P_{32})(1-P_{21}) = \rho_0(1+P_{32}-P_{21}-P_{32}P_{21}) \tag{6.36}$$

$$\rho_1{=}\rho_0(1+(1+P_{32})P_{21}) = \rho_0(1+P_{21}+P_{32}P_{21}) \tag{6.37}$$

Where $\rho_0 = \frac{D}{L_0}\frac{1}{D} = \frac{1}{L_0}$.

So

$$|\Delta\rho_{12}| = |\rho_1-\rho_2| = |\rho_0(P_{32}-2P_{21}-2P_{32}P_{21})| \tag{6.38}$$

$$|\Delta\rho_{23}| = |\rho_2-\rho_3| = |\rho_0(2P_{32}-P_{21}-P_{32}P_{21})| \tag{6.39}$$

$$|\Delta\rho_{13}| = |\rho_1-\rho_3| = |\rho_0(P_{32}+P_{21}+P_{32}P_{21})| \tag{6.40}$$

And we optimize.

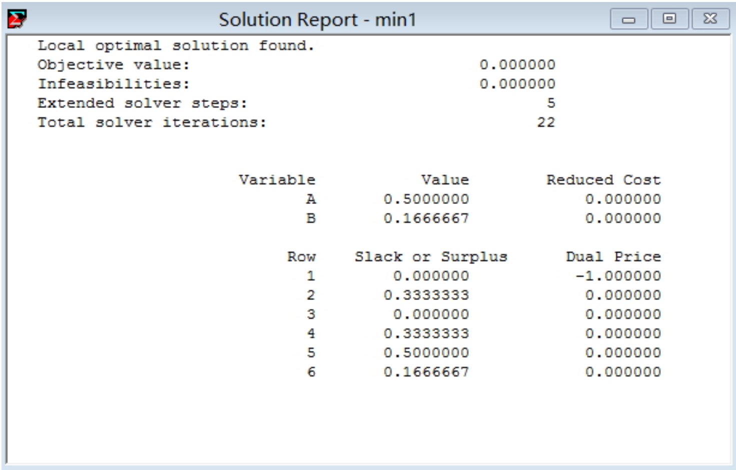
Let $P_{32} = A, P_{21} = B$, and

$$\min(A-2B-2AB)^2, \quad B \le A, \quad 0 < A \le \frac{1}{2}, \quad 0 < B \le \frac{1}{2} \tag{6.41}$$

$$\min(2A-B-AB)^2, \quad B \le A, \quad 0 < A \le \frac{1}{2}, \quad 0 < B \le \frac{1}{2} \tag{6.42}$$

$$\min(A+B+AB)^2, \quad B \le A, \quad 0 < A \le \frac{1}{2}, \quad 0 < B \le \frac{1}{2} \tag{6.43}$$

Using Lingo to solve these optimization respectively we get figure 6.9, figure 6.10, and figure 6.11:



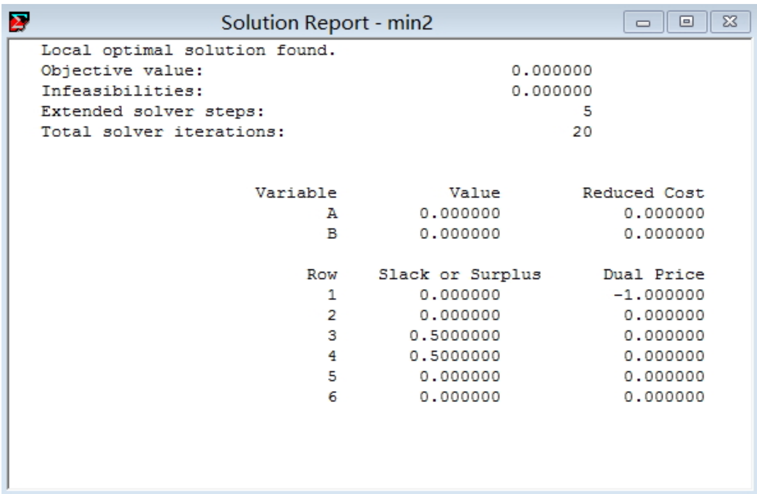
Solution Report - min1

```
Local optimal solution found.
Objective value:                          0.000000
Infeasibilities:                          0.000000
Extended solver steps:                           5
Total solver iterations:                        22

                 Variable        Value       Reduced Cost
                        A    0.5000000           0.000000
                        B    0.1666667           0.000000

                      Row  Slack or Surplus   Dual Price
                        1     0.000000          -1.000000
                        2    0.3333333           0.000000
                        3     0.000000           0.000000
                        4    0.3333333           0.000000
                        5    0.5000000           0.000000
                        6    0.1666667           0.000000
```

Figure 6.9: Lingo optimization 1



Solution Report - min2

```
Local optimal solution found.
Objective value:                          0.000000
Infeasibilities:                          0.000000
Extended solver steps:                           5
Total solver iterations:                        20

                 Variable        Value       Reduced Cost
                        A     0.000000           0.000000
                        B     0.000000           0.000000

                      Row  Slack or Surplus   Dual Price
                        1     0.000000          -1.000000
                        2     0.000000           0.000000
                        3    0.5000000           0.000000
                        4    0.5000000           0.000000
                        5     0.000000           0.000000
                        6     0.000000           0.000000
```

Figure 6.10: Lingo optimization 2

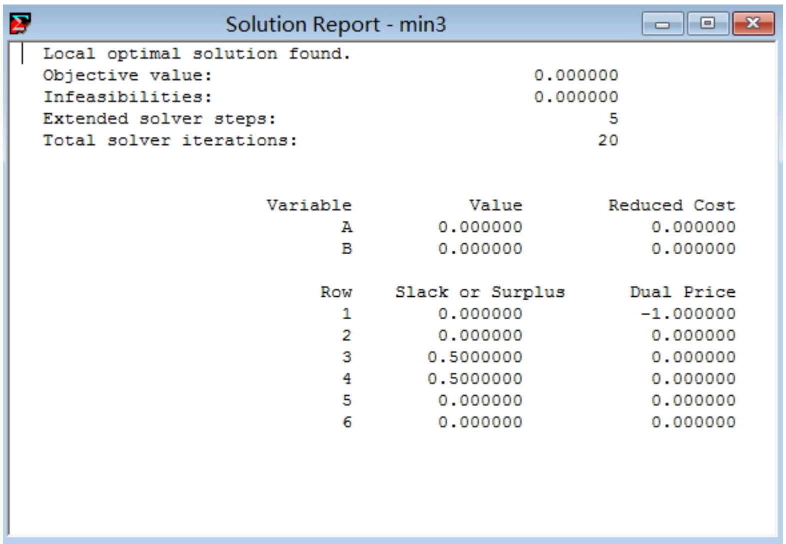


Figure 6.11: Lingo optimization 3

We see that when $P_{32} = 0.5, P_{21} = 0.1667$ or $P_{32} \to 0, P_{21} \to 0$ it is fairest. However, in the latter $x_2 = x_3 = D$ so the driver would have to merge twice in a row, and we have $P^2 << 0.5 \times 0.1667 = 0.08335$. So the former is a better choice.

Here $x_{32} = 0$, and with figure 6.12:

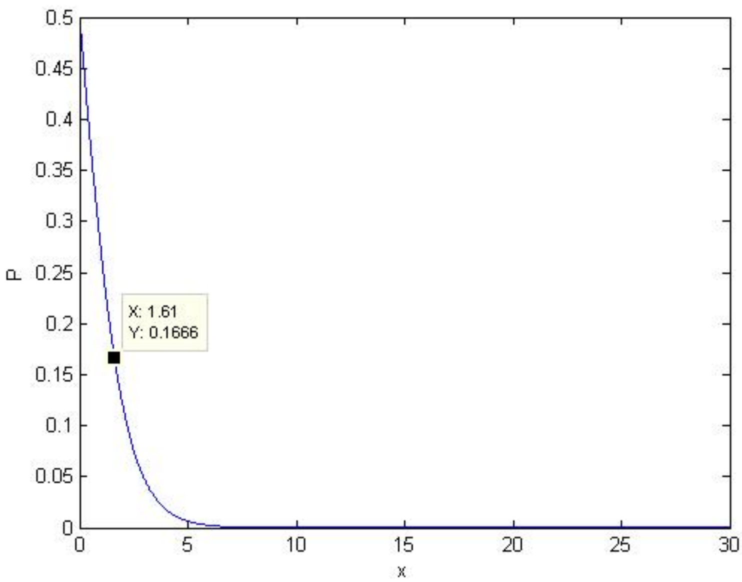


Figure 6.12: relationship between *P* and *x*, with x_{32}=0

We see that $x_2 = \frac{1.61}{30}D = 0.0537D$, which means that the driver in lane 3 merges immediately after seeing the sign and the driver in lane 2 merges at 0.0537*D* to make it fairest.

Once again, we have a problem.

7.2.4 The 3-to-1 merge: proposed merge model

Of course, we still have

$$\rho_3 = \rho_0(1-P_{32}) \tag{6.44}$$

$$\rho_2 = \rho_0(1+P_{32})(1-P_{21})P_z = \frac{1}{2}\rho_0(1+P_{32}-P_{21}-P_{32}P_{21}) \tag{6.45}$$

$$\rho_1 = \rho_0(1+(1+P_{32})P_{21})P_z = \frac{1}{2}\rho_0(1+P_{21}+P_{32}P_{21}) \tag{6.46}$$

Where $\rho_0 = \frac{D}{L_0}\frac{1}{D} = \frac{1}{L_0}$.

And,

$$|\Delta\rho_{12}| = |\rho_1 - \rho_2| = |\rho_0(P_{21} - \frac{1}{2}P_{32} + P_{32}P_{21})| \tag{6.47}$$

$$|\Delta\rho_{23}| = |\rho_2 - \rho_3| = |\rho_0(-\frac{1}{2} + P_{32} + \frac{1}{2}P_{21} + \frac{1}{2}P_{32}P_{21})| \tag{6.48}$$

$$|\Delta\rho_{13}| = |\rho_1 - \rho_3| = |\rho_0(-\frac{1}{2} + P_{32} + \frac{1}{2}P_{21} + \frac{1}{2}P_{32}P_{21})| \tag{6.49}$$

We let $P_{32} = A, P_{21} = B$, and solve:

$$\min(-\frac{1}{2}A + B + AB)^2, \quad 0 < A \le \frac{1}{2}, \quad 0 < B \le \frac{1}{2} \tag{6.50}$$

$$\min(-\frac{1}{2} + \frac{3}{2}A - \frac{1}{2}B - \frac{1}{2}AB)^2, \quad 0 < A \le \frac{1}{2}, \quad 0 < B \le \frac{1}{2} \tag{6.51}$$

$$\min(-\frac{1}{2} + A + \frac{1}{2}B + \frac{1}{2}AB)^2, \quad 0 < A \le \frac{1}{2}, \quad 0 < B \le \frac{1}{2} \tag{6.52}$$

Using Lingo we get figure 6.13, figure 6.14, and figure 6.15:

```
Solution Report - LINGO1

Local optimal solution found.
Objective value:                              0.1064926E-21
Infeasibilities:                               0.000000
Extended solver steps:                                5
Total solver iterations:                             28


                      Variable           Value        Reduced Cost
                             A     0.3228019E-04        0.000000
                             B     0.1613958E-04        0.000000

                           Row    Slack or Surplus      Dual Price
                             1        0.000000         -1.000000
                             2       0.4999677          0.000000
                             3       0.4999839          0.000000
                             4     0.3228019E-04        0.000000
                             5     0.1613958E-04        0.000000
```

Figure 6.13: Lingo optimization 4

```
Solution Report - LINGO1

Local optimal solution found.
Objective value:                              0.3752527E-20
Infeasibilities:                               0.000000
Extended solver steps:                                5
Total solver iterations:                             26


                      Variable           Value        Reduced Cost
                             A       0.3373712          0.000000
                             B     0.9057697E-02        0.000000

                           Row    Slack or Surplus      Dual Price
                             1        0.000000         -1.000000
                             2       0.1626288          0.000000
                             3       0.4909423          0.000000
                             4       0.3373712          0.000000
                             5     0.9057697E-02        0.000000
```

Figure 6.14: Lingo optimization 5

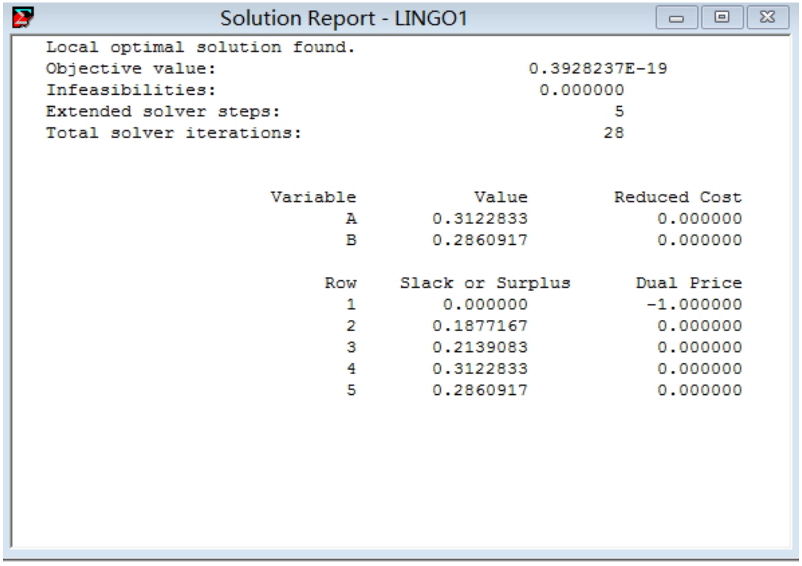


Figure 6.15: Lingo optimization 6

We can get three solutions, but when there are two merge sites, cars in lanes 1 or 2 merge at the lane closure, so the other merge site can't be two close to that (or else it's hardly another merge site), which is to say, the value of B cannot be too small (too close to 0). Therefore we choose $P_{32} = 0.3123, P_{21} = 0.2861$, and from figure 6.16:

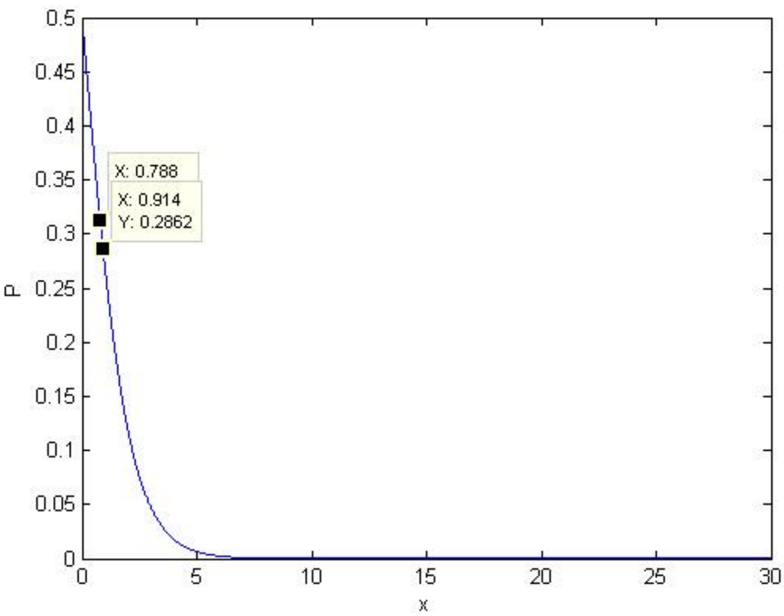


Figure 6.16: relationship between *P* and *x*, with P_{32}=0.3123, P_{21}=0.2861

The distance between the merge site and the sign is $x_3 = \dfrac{0.788}{30} D = 0.0263D$,

$$x_2 = \frac{0.914}{30} D = 0.0305D.$$

According to this model, cars in lane 3 should merge at 0.0263*D* and a third of cars in lane 2 should merge at 0.0305*D*. The rest of the cars wait for the zipper merge at the closure. This is more efficient than the "fairest" situation in the regular merge model.

The more easily implemented method: lane-2-and-3-cars merge left whenever possible and unmerged cars zipper-merge in the end. This is both most efficient and fairest and is consistent with the verbal analysis.

8 Third Scenario: Different Speed Limit

According to our models addressed previously, the strategy for merging is not dependent on the speed limit and only dependent to how you change lanes. So there is not much difference in your strategies if you are on a second road of 35 mph verses on a highway of 65 mph. You only have to change a little bit.

In the 2-lane merge case of step II, our recommendation is: at about 300 hundred feet ahead of the closure, right-lane drivers adjust to the speed of the left lane and prepare to merge if you are driving at a speed of 65 mph. If you are driving at 35 mph, the point you change lanes will be 168 feet before the merge point.

$$35 mile / hour = 51.3 feet / \sec ond \tag{6.53}$$

$$(51.3 ft / s)(3s) + 14 ft = 168 feet \tag{6.54}$$

If the speed limit is 25 mph, the changing-lane point will be 124 feet ahead of the closure.

In addition, since the safe distance will be shorter, it'll be slightly harder to squish into the left lane. However, this does not have much effect on the strategy used.

9 Sensitivity Analysis

The models used above neglected the parameter car density, ρ_j, when introducing the probability function, for simplicity. Here address the sensitivity of the effect of the car density.

We define the sensitivity function S as $S = \frac{e^{\rho}}{1} = e^{\rho} \quad (0 \le \rho \le \rho_j)$.

If $l = 14ft$, we have $\rho_j = \frac{1}{14} = 0.0714$, and the $S - \rho$ graph is shown in figure 6.17:

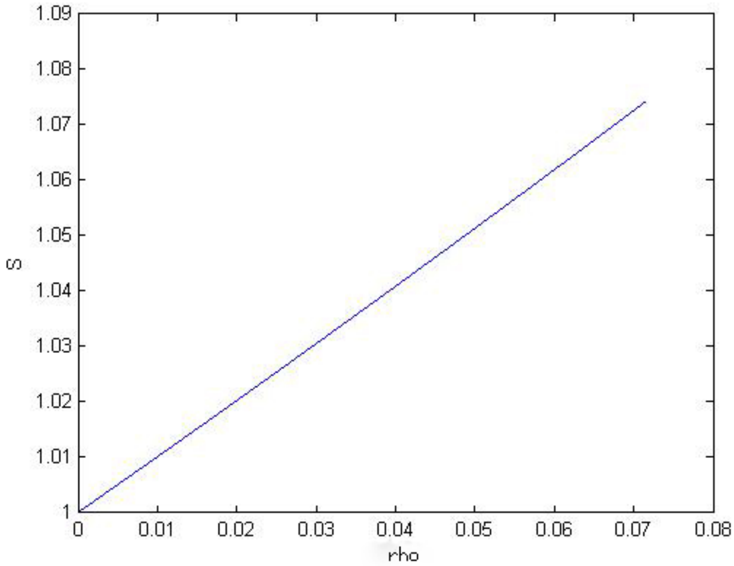


Figure 6.17: the sensitivity function *S*

From this graph we see that the divergence of e^{ρ} from its assumed value 1 increases with the car density. Which is to say, as ρ_j increases, the sensitivity of the model decreases. However, the value of S is still very close to 1, so the decrease in sensitivity is relatively small.

10 Guidelines for the DMV

We demonstrated that for the 2-to-1 lane merge, the "later zipper merge" is both more efficient and fairer than the currently prevalent practice, early merge. In order to promote the "later zipper merge," road signs need to be changed to direct automobile drivers to "use both lanes to the merge point." A sign saying "Zipper Merge" should be posted at about 300 feet from the lane closure so that the drivers know both lanes have equal rights and should take turns. Highway laws and driver education materials also need to be changed accordingly.

For 3-to-2 and 3-to-1 lane merge, we found that the current practice does fine. No changes need to be made.

11 Strengths and Weaknesses

Now we will analyze the strengths and weaknesses of each sub-model in our paper:

- **Strengths**

1. We used logical analysis along with mathematical computation throughout this paper, to ensure that the logic is verified while the mathematical model doesn't diverge from reality.

2. In the models we used variables as input (like D, v_0, l) instead of specific values, which makes our model apply to a number of different situations.

3. We searched on the Internet and found that though no one has built a mathematical model about it, many researchers have obtained results similar to ours. For example, Germany and Belgium already use the "later zipper merge" in their laws. [4]

- **Weaknesses**

1. We didn't create a computer simulation, which would have greatly increased our confidence in the model.

2. We only created a model for busy roads, and didn't focus much on other cases.

12 Possible Extensions

Given enough time, we could create an algorithm to simulate the conditions on a busy road to further verify our results. We could create models where the road isn't busy (where the distance between cars is greater than the safe distance) and analyze whether the later zipper merge is still most efficient and fairest. We could find how busy the road needs to be to make the later zipper merge more advantageous.

13 References

[1] *California Driver Handbook* [Edmund G. Brown Jr., Governor State of California; Brian P. Kelly, Secretary California State Transportation Agency; Jean Shiomoto, Director California Department of Motor Vehicles] page 57.

[2] www.smartmotorist.com/traffic-and-safety-guideline/maintain-a-safe-following-distance-the-3-second-rule.html. See also: *California Driver Handbook* [Edmund G. Brown Jr., Governor State of California; Brian P. Kelly, Secretary California State Transportation Agency; Jean Shiomoto, Director California Department of Motor Vehicles] page 57

[3] www.ask.com/vehicles/average-length-car-2e853812726d079d

[4] https://en.m.wikipedia.org/wiki/Late_merge See also: driveeuropenews.com/2013/11/09/what-is-the-zipper-method-designed-to-prevent/

14 Appendix: Letter to the DMV

Dear Director of the Department of Transportation:

We would like to propose a suggestion to improve the efficiency of our current practice regarding road merge. According to our study, the “later zipper merge” strategy is more efficient than the currently prevalent “early merge”. Germany and Belgium have already implemented this practice. We recommend that the US change our laws and driver education accordingly.

Since through traffic has the right-of-way according to most laws of US jurisdiction, when drivers see “right lane closed ahead” or “lane merge” signs on a two-lane highway, most move into the left lane fairly quickly. We call this an “early merge” strategy.

However, this practice can become complicating. Often, cars will form a long line in the left lane and leave the right lane empty. Some drivers will drive in the empty right lane, pass many “good and lawful” early drivers, and squeeze in at the very last moment at the closure. Oh! The injustice in this! The already merged drivers might get irritated and blow their horns. Behold: road rage. Even worse, they insist upon their lawful rights and block the drivers on the right lane from merging. This might lead to accidents. The right-lane driver might also earn a “failure to yield” or “unsafe merge” citation because it is always their fault in our current law system.

This situation could be improved if we adapted a “later zipper merge” strategy. Of course, if you are the only car on a major highway, it doesn’t matter where you merge. This strategy shows its advantage only on a busy two-lane road. The procedure for an individual driver is like this:

1) You stay in your own lane and maintain your current speed for as long as possible.

2) At about 300 feet from the closure, adjust to the speed of the left lane and prepare to merge (if the speed limit is 65 mph).

3) Drivers leave room for the car in the adjacent lane to merge.

4) The drivers in both lanes take turns to enter the through lane like teeth on a zipper while maintaining their current speed.

On the other hand, when three-lane roads merge into two or one, drivers should use another approach: merge left whenever possible, and zipper-merge 300ft from the closure if they don’t manage that. This is very similar to the current practice of most drivers.

In addition, for slower roads the merge may be postponed. For example, on 35mph-roads, the merge point should be 168ft from the lane closure.

Our math model demonstrates that these strategies are not only efficient but also fair.

It will reduce travel time, congestion, and road rage. The detailed reasoning is in our math model thesis enclosed.

If you agree that our new strategy is feasible, several changes need to be made to our current highway system.

For our Federal Highway Administration or construction department, they should use signs to direct automobile drivers at two-lane merges to “use both lanes to the merge point.” At about 300 feet (or 168ft for slow roads) ahead of the lane closure, a sign should be posted saying “Zipper Merge,” so that the drivers know both lanes have equal rights and should take turns.

We should incorporate the “later zipper merge” into our highway laws and driver education materials to promote efficient, fair, and safe driving.

Your feedback is highly appreciated. Our email: XXXXXX.

Sincerely,
The Mad Math Group

Nov 15, 2015